ISRAEL JOURNAL OF MATHEMATICS **139** (2004), 177–181

A THEOREM OF NAGATA AND MUKAI/SAKAI VIA FOURIER-MUKAI TRANSFORM

BY

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ABSTRACT

We give a new proof of a theorem of Nagata and Mukai/Sakai about line subbundles of high degrees of vector bundles on curves. The main tool is the Fourier–Mukai transformation on the Jacobian of the curve.

1. Introduction

Let k be an algebraically closed field and let C/k be an irreducible smooth projective curve of genus g. For a vector bundle $\mathcal{E}$ over C we consider its Euler characteristic $\chi(\mathcal{E}) := \dim_k H^0(C, \mathcal{E}) - \dim_k H^1(C, \mathcal{E})$. The Riemann–Roch theorem gives a formula $\chi(\mathcal{E}) = \operatorname{rank}(\mathcal{E})(1-g) + \deg(\mathcal{E})$. Then we have the following theorem of Mukai and Sakai ([M-S]):

THEOREM 1.1 (Nagata, Mukai/Sakai): *Every vector bundle $\mathcal{E}$ over C with $\chi(\mathcal{E}) > -g$ contains a line bundle of nonnegative degree.*

The case $\operatorname{rank}(\mathcal{E}) = 2$ was first proved by Nagata [Na]. Later on, Mukai and Sakai proved the general result using Grothendieck's Quot-scheme as an important tool. Actually, they proved a more general result about the existence of subbundles of high degree.

We give another proof of this theorem using the Fourier–Mukai transformation on the Jacobian of the curve.

ACKNOWLEDGEMENT: The author thanks Jochen Heinloth and an anonymous referee for helpful remarks, which improved the exposition.

Received September 26, 2002

2. Fourier–Mukai transform

We start with some notation and recall the definition and main properties of the Fourier–Mukai transformation on abelian varieties.

Let X be a g-dimensional abelian variety and $\widehat{X}$ its dual abelian variety. The product $X \times \widehat{X}$ carries a normalized Poincaré bundle $\mathcal{P}$. Here "normalized" means that both $\mathcal{P}|_{X\times\{0\}}$ and $\mathcal{P}|_{\{0\}\times\widehat{X}}$ are trivial. We call pr_1 resp. pr_2 the projection to the first resp. second factor. We denote by $D(X)$ the derived category of the category of $\mathcal{O}_X$-modules, and $D^b_{coh}(X)$ its full subcategory consisting of complexes with bounded, coherent cohomology. For a complex A we define $A[n]$ to be the complex shifted by n to the left, i.e. $A[n]_i = A_{i+n}$. Let m: $X \times X \to X$ be the group law on X, pr_1 and pr_2 the projections. For $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ we have the "exterior" tensor product $\mathcal{F} \boxtimes \mathcal{G} := pr_1^*\mathcal{F} \otimes pr_2^*\mathcal{G}$ and the Pontrjagin product $\mathcal{F} * \mathcal{G} := m_*(\mathcal{F} \boxtimes \mathcal{G})$. Then $*$ is a bifunctor from $\mathrm{Mod}(X) \times \mathrm{Mod}(X) \to \mathrm{Mod}(X)$. We denote its derived functor by $\overset{R}{*}$. Building iterated products $\mathcal{F}^{*r} := \mathcal{F} \overset{R}{*} \cdots \overset{R}{*} \mathcal{F}$ and $\mathcal{F}^{\boxtimes r} := \mathcal{F} \boxtimes \cdots \boxtimes \mathcal{F}$ (r-times) we have isomorphisms $\mathcal{F}^{*r} \simeq \mathbf{R}m_*(\mathcal{F}^{\boxtimes r})$. Again m: $X^r \to X$ denotes the group law. For morphisms f: $X_1 \to Y_1$, g: $X_2 \to Y_2$ and $\mathcal{O}_{X_1}$-module $\mathcal{F}$, $\mathcal{O}_{X_2}$-module $\mathcal{G}$, there is a Künneth formula: $\mathbf{R}(f \times g)_*(\mathcal{F} \boxtimes \mathcal{G}) = \mathbf{R}f_*(\mathcal{F}) \boxtimes \mathbf{R}g_*(\mathcal{G})$. The translation by a k-rational point x is denoted by T_x.

We define the functor $\mathbf{R}\mathcal{S}$ from $D^b_{coh}(X)$ to $D^b_{coh}(\widehat{X})$ by

$$\mathbf{R}\mathcal{S}(?) = \mathbf{R}pr_{2*}(pr_1^*(?) \overset{L}{\otimes} \mathcal{P})$$

and also the functor $\mathbf{R}\widehat{\mathcal{S}}$ from $D^b_{coh}(\widehat{X})$ to $D^b_{coh}(X)$ by

$$\mathbf{R}\widehat{\mathcal{S}}(?) = \mathbf{R}pr_{1*}(pr_2^*(?) \overset{L}{\otimes} \mathcal{P}).$$

They are called the Fourier–Mukai and inverse Fourier–Mukai transform, respectively.

We list the main properties of the Fourier-Mukai transformation ([Mu]):

(a) $\mathbf{R}\widehat{\mathcal{S}} \circ \mathbf{R}\mathcal{S} \simeq (-1_X)^*[-g]$, $\mathbf{R}\mathcal{S} \circ \mathbf{R}\widehat{\mathcal{S}} \simeq (-1_{\widehat{X}})^*[-g]$.

In particular, $\mathbf{R}\mathcal{S}$ and $\mathbf{R}\widehat{\mathcal{S}}$ define equivalences of categories.

(b) $\mathbf{R}\mathcal{S}(\mathcal{F} \overset{R}{*} \mathcal{G}) \simeq \mathbf{R}\mathcal{S}(\mathcal{F}) \overset{L}{\otimes} \mathbf{R}\mathcal{S}(\mathcal{G})$, $\mathbf{R}\mathcal{S}(\mathcal{F} \overset{L}{\otimes} \mathcal{G}) \simeq \mathbf{R}\mathcal{S}(\mathcal{F}) \overset{R}{*} \mathbf{R}\mathcal{S}(\mathcal{G})[g]$.

(c) $\mathbf{R}\mathcal{S} \circ T_x^* \simeq (? \overset{L}{\otimes} \mathcal{P}_{-x}) \circ \mathbf{R}\mathcal{S}$, $\mathbf{R}\mathcal{S} \circ (? \overset{L}{\otimes} \mathcal{P}_{\hat{x}}) \simeq T_{\hat{x}}^* \circ \mathbf{R}\mathcal{S}$.

Iterating the isomorphism in (b) we get an isomorphism $\mathbf{R}\mathcal{S}(\mathcal{F}^{*r}) \simeq \mathbf{R}\mathcal{S}(\mathcal{F})^{\otimes r}$. Going through the proof of identity (b) ([Mu], p. 160), one checks that this isomorphism could be chosen to commute with the natural permutation operation of the symmetric group S_r on both sides.

The following lemma is certainly well known.

LEMMA 2.1: *Let $\mathcal{L}$ be a non-degenerate line bundle on X. Then its Fourier–Mukai transform is a shifted vector bundle, i.e. there is a vector bundle $\mathcal{V}$ on $\widehat{X}$ and an integer n with $\mathbf{R}\mathcal{S}(\mathcal{L}) \simeq \mathcal{V}[n]$.*

Proof: By the vanishing theorem the cohomology $H^*(X, \mathcal{L})$ vanishes in all degrees but one, called the *index*, for which it is nonzero ([Mum], 150). From the Riemann–Roch theorem and an "index theorem", it follows that this index i_0 and the dimension of $H^{i_0}(X, \mathcal{L})$ depend only on the image of $\mathcal{L}$ in the Néron–Severi group ([Mum], p. 150 and p. 155). By base change theorems $\mathbf{R}^i\mathcal{S}(\mathcal{L})$ vanishes for $i \neq i_0$ and $\mathbf{R}^{i_0}\mathcal{S}(\mathcal{L})$ is locally free (e.g. [Mum], §5). ∎

3. Proof of the theorem

Let $\mathcal{A}$ be the Albanese variety of C. Fixing a closed point on C we get an embedding i: $C \hookrightarrow \mathcal{A}$. This embedding induces an isomorphism of their Picard varieties of algebraically trivial line bundles: i^*: $\widehat{\mathcal{A}} \xrightarrow{\sim} Pic^0(C)$. Fourier–Mukai transformation gives an equivalence of categories $R\mathcal{S}$: $D^b_{coh}(\mathcal{A}) \to D^b_{coh}(Pic^0(C))$. Note that the complex $R\mathcal{S}(i_*\mathcal{E})$ has nontrivial cohomology only in degrees $0, 1$. Before we study this complex in more detail we need a preparation. We denote by $\overline{m}$: $C^r \to \mathcal{A}$ the restriction of the group law.

PROPOSITION 3.1: *For any integer $r > 0$ we have $(i_*\mathcal{E})^{*r} \simeq \mathbf{R}\overline{m}_*(\mathcal{E}^{\boxtimes r})$.*

Proof: Let i^r: $C^r \hookrightarrow \mathcal{A}^r$ be the product of copies of i. By the Künneth formula we have $(i_*\mathcal{E})^{\boxtimes r} \simeq (i^r)_*\mathcal{E}^{\boxtimes r}$. Applying $\mathbf{R}m_*$ gives the claim. ∎

Proof of Theorem 1.1: Assume that the theorem is false. Then $H^0(C, \mathcal{E} \otimes \mathcal{P}_{\hat{x}}) = \{0\}$ for all closed points $\hat{x} \in Pic^0(C)$ and $\dim_k H^1(C, \mathcal{E} \otimes \mathcal{P}_{\hat{x}}) = -\chi(\mathcal{E})$ is independent of $\hat{x}$. In particular, $\chi(\mathcal{E}) \leq 0$. Applying base change theorems we get that $\mathcal{V} := R^1\mathcal{S}(i_*\mathcal{E})$ is a vector bundle of rank $r := -\chi(\mathcal{E})$ and that $R^0\mathcal{S}(i_*\mathcal{E})$ vanishes, i.e. $R\mathcal{S}(i_*\mathcal{E}) = \mathcal{V}[-1]$. For $r = 0$ we have $\mathcal{V} = 0$. Since $R\mathcal{S}$ is an equivalence of categories this implies $i_*\mathcal{E} = 0$, a contradiction. Therefore, in the following we assume $r > 0$. We define a locally free sheaf $\mathcal{H}$ by the following exact sequence:

$$0 \to \mathcal{H} \to \mathcal{V}^{\odot r} \to \det \mathcal{V} \to 0.$$

Note that the symmetric group S_r acts on all these vector bundles by permuting the tensor factors and that $\epsilon := \sum_{\sigma \in S_r} \operatorname{sign}(\sigma)\sigma$ induces the zero map on $\mathcal{H}$.

Applying $\mathbf{R}^*\widehat{\mathcal{S}}$ we get a long exact sequence

$$\cdots \to \mathbf{R}^m\widehat{\mathcal{S}}(\mathcal{H}[-r]) \to \mathbf{R}^m\widehat{\mathcal{S}}(\mathcal{V}^{\otimes r}[-r]) \to \mathbf{R}^m\widehat{\mathcal{S}}(\det \mathcal{V}[-r]) \to \cdots.$$

By property (b) we know $\mathbf{R}\mathcal{S}((i_*\mathcal{E})^{*r}) \simeq \mathcal{V}^{\otimes r}[-r]$. Hence by property (a) and Proposition 3.1 we have for the middle term of the above sequence

$$\mathbf{R}^m\widehat{\mathcal{S}}(\mathcal{V}^{\otimes r}[-r]) \simeq H^m((-1_{\mathcal{A}})^*((i_*\mathcal{E})^{*r})[-g]) \simeq (-1_{\mathcal{A}})^*\mathbf{R}^{m-g}\overline{m}_*(\mathcal{E}^{\boxtimes r}).$$

Taking $m = g$ in the above sequence we get

$$\mathbf{R}^g\widehat{\mathcal{S}}(\mathcal{H}[-r]) \to (-1_{\mathcal{A}})^*\overline{m}_*(\mathcal{E}^{\boxtimes r}) \xrightarrow{\phi} \mathbf{R}^g\widehat{\mathcal{S}}(\det \mathcal{V}[-r]).$$

We will prove the following two properties for the map ϕ:

CLAIM 1: $\phi \neq 0$.

CLAIM 2: *The target space of* ϕ, *i.e.* $\mathbf{R}^g\widehat{\mathcal{S}}(\det \mathcal{V}[-r])$, *is locally free.*

Since we are assuming $r < g$, the generic fibre of $(-1_{\mathcal{A}})^*\overline{m}_*\mathcal{E}^{\boxtimes r}$ is zero. Hence there is no morphism to a locally free sheaf but the zero map. This is a contradiction to our claims.

Proof of Claim 1: The summation map $\overline{m}$: $C^r \to \mathcal{A}$ factorizes through the symmetric product $C^{(r)}$ and the induced map $C^{(r)} \to \mathcal{A}$ is birational to its image W^r([Mi], Theorem 5.1). Therefore, there is a non-empty open subset $U \subseteq W^r$ such that $\overline{m}$: $\overline{m}^{-1}(U) \to U$ is an unramified Galois covering with Galois group S_r. Note that any $(x_1, \ldots, x_r) \in \overline{m}^{-1}(U)$ has pairwise distinct components. Let $\underline{x} = (x_1, \ldots, x_r) \in \overline{m}^{-1}(U)$ and set $y = \overline{m}(\underline{x})$. We get $(\overline{m}_*(\mathcal{E} \boxtimes \cdots \boxtimes \mathcal{E}))_y = \bigoplus_{\sigma \in S_r} \mathcal{E}_{x_{\sigma(1)}} \otimes \cdots \otimes \mathcal{E}_{x_{\sigma(r)}}$. Obviously ϵ does not induce the zero map on these fibres. Since, on the other hand, ϵ induces the zero map on $\mathbf{R}^g\widehat{\mathcal{S}}(\mathcal{H}[-r])$ by functoriality, the map ϕ cannot vanish at fibres over U. Claim 1 follows. ∎

Proof of Claim 2: For an appropriate $a \in \mathcal{A}(k)$ the intersection $U \cap (a + C)$ is non-empty and open in $a + C$. By the proof of Claim 1 we know that the fibres of $\mathbf{R}^g\widehat{\mathcal{S}}(\det \mathcal{V}[-r])$ are non-vanishing at all points in U. So the restriction of $\mathbf{R}^g\widehat{\mathcal{S}}(\det \mathcal{V}[-r])$ to the curve $a + C$ has a non-vanishing generic fibre, say of rank R. Since the restriction map $Pic^0(\mathcal{A}) \xrightarrow{\sim} Pic^0(a + C)$ is an isomorphism, every line bundle $\mathcal{L} \in Pic^0(\mathcal{A})(k)$ with $\mathbf{R}\widehat{\mathcal{S}}(\det \mathcal{V}[-r]) \overset{L}{\otimes} \mathcal{L} \simeq \mathbf{R}\widehat{\mathcal{S}}(\det \mathcal{V}[-r])$ has to be of order at most R. In particular, there are only finitely many of them. Thus by property (c) $\det \mathcal{V}$ is a non-degenerate line bundle. Then by Lemma 2.1, $\mathbf{R}^g\widehat{\mathcal{S}}(\det \mathcal{V}[-r]) \simeq \mathbf{R}^{g-r}\widehat{\mathcal{S}}(\det \mathcal{V})$ is locally free, which is Claim 2. ∎

The proof given above obviously generalizes to higher dimensional subvarietes of abelian varieties, but under heavy conditions. The author does not know about their value. To be more precise, I formulate the surface case, not attempting greatest generality (changing notation).

THEOREM 3.2: *Let C be a smooth irreducible proper surface with canonical line bundle ω_C, let $\mathcal{E}$ be a vector bundle on C, let $\mathcal{A}$ be an abelian variety and let $i\colon C \to \mathcal{A}$ be a morphism. If $r := -\chi(\mathcal{E}) > 0$ we assume the following:*

(a) *The induced map $i^{(r)}\colon C^{(r)} \to \mathcal{A}$ is generically finite to its image.*

(b) *The restriction map ${i^{(r)}}^*\colon Pic^0(\mathcal{A}) \to Pic^0(C^{(r)})$ has finite kernel.*

(c) $2r < \dim \mathcal{A}$.

Then there is a line bundle $\mathcal{L} \in Pic^0(\mathcal{A})$ such that either $\mathrm{Hom}_{\mathcal{O}_C}(i^*\mathcal{L}, \mathcal{E}) \neq 0$ *or* $\mathrm{Hom}_{\mathcal{O}_C}(\mathcal{E}, i^*\mathcal{L} \otimes \omega_C) \neq 0$.

We note that the conditions (a)–(c) imply that i is generically finite to its image and that $\mathcal{A}$ is isogenous to a factor of the Albanese variety of C. For the proof of Theorem 3.2 one modifies the proof of Theorem 1.1 in the following way: We substitute $i_*\mathcal{E}$ by $\mathbf{R}i_*\mathcal{E}$. This does not affect the further proof. For the proof of Claim 2 we remark that there is the concept of the determinant of a perfect complex ("Knudsen–Mumford determinant"). In particular, we can speak about the determinant of any coherent sheaf on a regular variety. In this situation and for a coherent sheaf $\mathcal{Q}$ of generic rank r and line bundle $\mathcal{L}$ we have the formula $\det(\mathcal{Q} \otimes \mathcal{L}) = \det \mathcal{Q} \otimes \mathcal{L}^{\otimes r}$. We apply this to ${i^{(r)}}^* \mathbf{R}^g \widehat{\mathcal{S}}(\det \mathcal{V}[-r])$, which has non zero generic rank. From this and assumption (b), Claim 2 follows.

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